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Prime–index parametrization for total neutrino-nucleon cross sections and *pp* cross sections

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Abstract

A prime number based parametrization for total neutrino-nucleon cross section is presented. The method employs the relation between prime numbers and their indices to reproduce neutrino cross sections for neutrino energies from the MeV to the PeV regions where experimental data are available. This prime–index relation provides estimates of the neutrino-nucleon cross sections valid across many decades of neutrino energy scales. The PeV data are from the recently published astrophysical $\nu_\mu + \bar{\nu}_\mu$ rates in the IceCube detector as well as neutrino-nucleon cross section measurements. A similar method has been employed for high energy *pp* cross sections which explains the $(\ln s)^2$ parametrization first proposed by Heisenberg.

Keywords: neutrino cross section, *pp* cross section, prime numbers, particle astrophysics

(Some figures may appear in colour only in the online journal)

1. Introduction

Neutrino-nucleon and/or neutrino-nucleus cross section experiments are divided into three categories; low, medium and high energies. The choice of energy regions is motivated by neutrino production sources. The low energy regime includes reactor, geoneutrinos and supernova (SN) neutrinos constituting neutrinos below $\approx$10 MeV for reactor and geoneutrinos and below $\approx$60 MeV for SN neutrinos. Above $\approx$10 MeV and below $\approx$350 GeV neutrinos are produced in accelerators, where the energy spectra and fluxes as well as the flavor of neutrinos are known. In this medium energy region, there is also a contribution from the atmospheric neutrinos which extends to 100 TeV. At energies above 100 TeV, neutrinos are produced predominantly by astrophysical sources, recently observed by the IceCube experiment [1].

Theoretical approaches to cross section calculations also follow the above energy classifications, primarily because of the momentum transfers involved in the weak interaction processes. At low energies, in the MeV region, the usual Shell Model, Random Phase Approximation (RPA), Quasiparticle Random Phase Approximation (QRPA) or Effective Field Theory (EFT) calculations are usually performed [2]. RPA, QRPA and EFT calculations possess varying degrees of success depending on the target nucleus, where the finer aspects of nuclear structure effects play a significant role. In the medium energy region, Impulse Approximation folded in with a Fermi Gas Model, or Spectral Model calculations are often used to reproduce the experimental data [2]. At higher neutrino energies, above 0.1 TeV, Parton Model is mainly utilized which we refer to as the Standard Model (SM) [3]. A detailed review article of neutrino-nucleon/nucleus cross sections across energy scales with description of various calculation methods at different energies and comparison with experimental data was done by Formaggio and Zeller [4].

The monotonically increasing behavior of neutrino cross section measurements was the genesis for the low energy approximation of Vogel and Beacom [5]. This approximation includes first order corrections in $\varepsilon = E_\nu/m_p$ and provides an estimate at $\bar{\nu}_e$ energies below $E_{\bar{\nu}} < 60$ MeV.

$$\sigma \approx 9.53 \times 10^{-44} \frac{p_e E_e}{\text{MeV}^2}\ \text{cm}^2, \tag{1}$$

where $E_e = E_\nu \pm \Delta$ for $\bar{\nu}_e$ and ν_e, and $\Delta = M_n - M_p$ represent the neutron–proton mass difference.

Strumia and Vissani [6] have derived the following approximation shown in equation (2), that agrees well with their full calculations, below $E_\nu < 300$ MeV,

$$\sigma(\bar{\nu}_e p) \approx 10^{-43}\ \text{cm}^2\ p_e E_e E_\nu^{-0.07056+0.02018 \ln E_\nu - 0.001\,953 \ln^3 E_\nu}, \tag{2}$$

where energies are in MeV.

In the energy range of GeV to TeV, a linear energy dependence for neutrino and antineutrino nucleon cross sections of $\sigma = 0.677E$ and $\sigma = 0.334E$ are derived from the measured cross sections [7].

A rough estimate for νN charged current (CC) cross sections in ultra high energies, with $E > 10^6$ GeV and $E < 10^{12}$ GeV, which is well above the GZK limit, is provided by the following approximation [8],

$$\sigma(\bar{\nu}_e p) \approx 5.53 \times 10^{-36}\ \text{cm}^2\ E_\nu^{0.363}, \tag{3}$$

where E_ν is in GeV.

2. Primes and their Indices

In this paper, we propose to estimate the total neutrino-nucleon cross sections by using the relation between prime numbers and their indices. Prime numbers are a class of integers that are only divisible by themselves and one. By this definition, the number one itself is not considered a prime number. Some prime numbers and their indices are shown in table 1.

Mathematically, it is a formidable task to determine if a number is prime. Finding primes is usually done by the use of the *sieve theory* which is designed to count, or to estimate the size of sets of integers. The *sieve theory* also is used to sift out a set of prime numbers up to

Table 1. Some prime numbers with their positional indices.

Index	Prime
1	2
2	3
.	.
8	19
.	.
19	67
.	.
114	619
.	.
619	4567

some desired index. Note, we refer to the number in the left column of table 1 as the index of a prime. The index of a prime is simply the prime numbers ascending positional rank. There is an asymptotic relation between primes and their indices. This is due to *Gauss*'*s* prime number theorem that relates a prime to its index through the following relation;

$$\pi(p) \sim \frac{p}{\ln p}. \tag{4}$$

Gauss later modified equation (4) to;

$$\pi(p) \sim \int_2^p \frac{\mathrm{d}x}{\ln x}. \tag{5}$$

Gauss'*s* relation relates the index and its corresponding prime in an asymptotic manner and the index and the prime approach the actual values as they tend to infinity. Conversely, the prime number theorem is equivalent to stating that the *i*th prime number is

$$p_{\pi(p)} \sim \pi(p)\ln\pi(p). \tag{6}$$

3. Results and discussions

In this paper, we use relations between primes and their indices and twin primes (a pair of primes which differ by 2, such as 11, 13 or 107, 109) and their indices to estimate total CC and neutral current (NC) neutrino-nucleon cross sections and total *pp* cross sections. Note, these estimates are empirically driven and not to be considered as a replacement for the above-mentioned physics-based approaches.

As mentioned above, we utilize the relation between a given prime and its corresponding index to arrive at an estimate of the total cross section for the CC and NC neutrino-nucleon scattering. The method takes the index of the prime as the energy and the prime itself then provides the cross section for that energy. We assign the index to be in units of MeV. Note, the choice of MeV is made because it is of the order of the neutron–proton mass difference or of the nuclear binding energy and is maintained regardless of the energy region. The cross sections are then expressed in units of 0.70 of $10^{-42}\,\mathrm{cm}^2$ or in units of 0.70 *ato-barn* (1 $ab = 10^{-18}$ *barn*) for ν_eN CC cross sections. For example, at a neutrino energy of 19 MeV which is the index of the prime number 67, the cross section is 0.70×67 or $\approx 47\,ab$, i.e. $\approx 47 \times 10^{-42}\,\mathrm{cm}^2$. A two-column look-up table such as table 1 may be generated where the

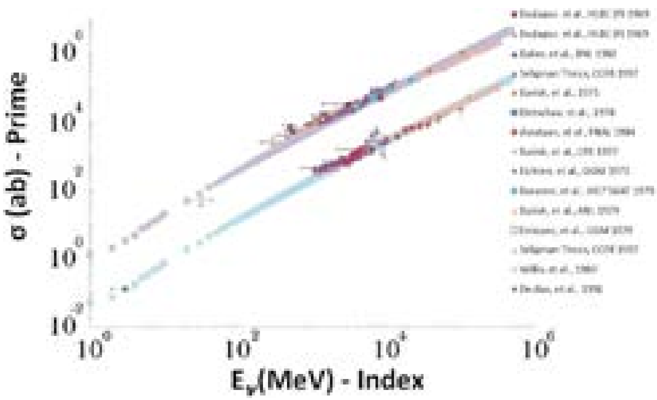

Figure 1. The visible energy distribution for neutrinos and antineutrinos on various nuclear targets. Note the data represents total neutrino and antineutrino reaction cross sections with an isoscalar nucleon. The antineutrino cross sections have been multiplied by 0.1 for easier visualization. The corresponding prime number distributions have been multiplied by 0.70 (blue bell squares) for neutrinos and by 0.26 for antineutrinos (dark sky blue squares).

first column is the index of the prime and it represents the neutrino energy in MeV and the second column is the prime number multiplied by 0.70. This represents the neutrino-nucleon cross sections in *ab*. For antineutrinos, the second column is multiplied by 0.26 to produce the cross sections also in *ab*. Note, the prime–index method provides a quick and accurate estimate of the neutrino-nucleon cross sections over many decades of energy scales and hence a reliable estimate for interaction rates in various neutrino experiments.

Cross section estimates can also be obtained from equation (6) where the index i is the energy in MeV and its corresponding prime, p, represents the un-normalized cross sections in *ab*. This method introduces a 22% error at low energies and about 7% error at high energies when compared to actual primes and their indices.

In figure 1 the flux-averaged cross sections for many accelerator produced neutrinos and antineutrinos reacting on various targets are shown [9][1]. Note, the values of the antineutrino cross sections data are multiplied by 0.1 for ease of visualization.

4. IceCube high energy astrophysical neutrinos

The IceCube collaboration published the observation of astrophysical muon neutrinos in the IceCube Detector [10]. The muons observed are clear indications of ν_μ and $\bar{\nu}_\mu$ CC interactions in the IceCube detector. The rate for these muon neutrinos have been compared with the SM calculations of Cooper-Sarkar *et al*, [11]. The prime–index method is also compared with those of [11]. Table 2 shows these total ν_μ CC cross sections $\geqslant$20 TeV and their ratios. Note, in obtaining the cross sections with the prime–index method, the average of the normalization for ν_μ and $\bar{\nu}_\mu$ CC cross sections of 0.70 and 0.26, i.e. 0.48 was used.

[1] The data points were extracted by K S Kuzmin, V V Lyubushkin and V A Naumow from digitization of published figures. Some of the data are also in the Durham Reaction Database. The digitization does not exactly coincide with those of Durham. The goal was to minimize the disagreements, but sometimes it was not possible. In these cases preference was given to the digitization, except in cases where the Durham data were corrected by authors. Extractions were tested when possible with several versions of the figures in articles, preprints, conference reports, etc.

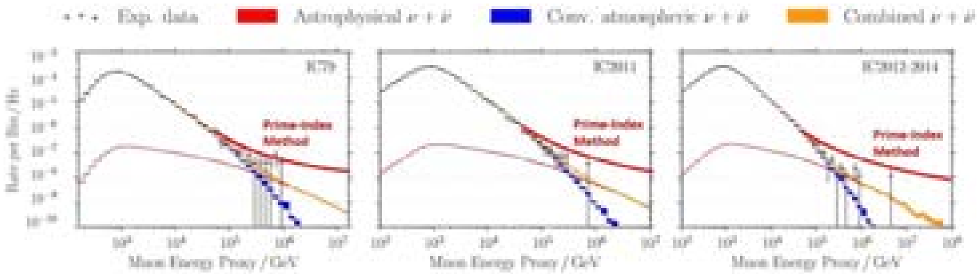

Figure 2. The proposed prime–index method superimposed on a figure showing the experimental data from the published IceCube data of reference [10].

Table 2. Comparison of prime–index cross sections with those of reference [11].

Energy/GeV	σ_T/pb (SM)	σ_T/pb (prime–index)	Ratio
20000	77	177	2.30
50000	140	469	3.35
100000	210	975	4.64
200000	310	2036	6.57
500000	490	5292	10.80
1×10^6	690	10978	15.91
2×10^6	950	22707	23.90
5×10^6	1400	59078	42.20
1×10^7	1900	121830	64.12

Table 3. Comparison of IceCube data above 0.5 PeV with prime–index rate and with the SM.

Energy/PeV	IceCube data	prime–index method	SM
>0.5	9 ± 3	11 ± 3.3	1 ± 1

The prime–index cross sections and the ratio to the SM values were used to calculate the expected rate for $\nu_\mu + \bar{\nu}_\mu$ in the IceCube detector. Figure 2 shows the prime–index rate prediction.

The number of events observed above 0.5 PeV where the neutrinos are expected to be predominantly astrophysical are shown in table 3. For comparison, the number of events expected from the prime–index method and the SM are also listed. Even with the limited number of events, there is strong evidence that the prime–index method is consistent with the observed number of events in the IceCube detector.

Recent IceCube analysis of neutrino and antineutrino cascade events for CC and NC at high energies provide data to be compared with the prime–index method. The ratio of the CC cross section to the CC + NC cross sections can be written as;

$$\frac{\sigma_{\mathrm{CC}}}{\sigma_{\mathrm{CC}} + \sigma_{\mathrm{NC}}} \approx 0.7. \tag{7}$$

The normalization factors used in this paper for high energy astrophysical neutrinos are 0.48 and 0.69 for the data in figures 2 and 3, respectively. These factors assume that the data were

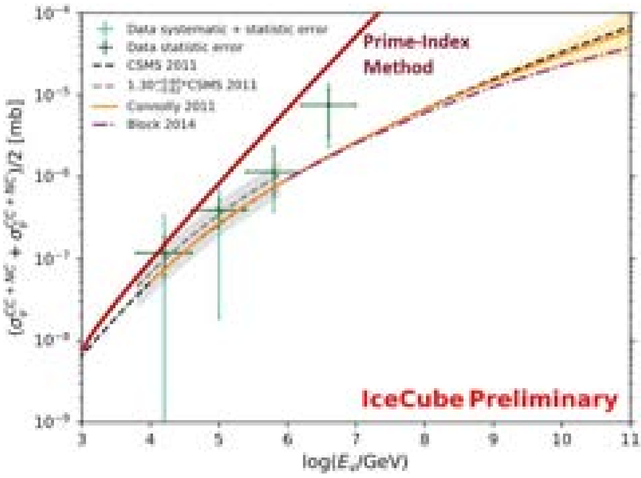

Figure 3. The proposed prime–index method superimposed on a figure showing the CC and NC experimental data from the IceCube experiment. The prime–index curve has a normalization factor of 0.69.

obtained with various neutrino and antineutrino flavors have roughly the same flux. This is only true for more statistically significant measurements. The abovesaid overall measurements are obtained with limited statistics and each energy bin contain even less events. Hence, the use of the above normalization factors should be only considered as a first step approximation until further analyses of the IceCube data provide a larger dataset.

At high neutrino energies, above 0.5 PeV, our proposed method when compared to those obtained from the SM begin to diverge by an order of magnitude and increases as shown in table 2. At 10^{12} GeV, well above the GZK limit, cross sections obtained by the prime–index method are of the order 100 mb, 10^6 times larger than those predicted by SM. The proposed 10 km^3 upgrade to the IceCube detector, [12] could provide the ideal laboratory for this investigation. Also, further analysis of the IceCube data is highly anticipated.

5. Total *pp* cross section

The total square of energy s in the Center of Mass (CM) frame can be written as;

$$s = 2M_pE + {M_p}^2. \tag{8}$$

At high energies where $E \gg M_p$, we may neglect the term ${M_p}^2$;

$$\sqrt{s} = \sqrt{2M_pE}. \tag{9}$$

The suggestion, originally by Heisenberg [13], proposed a universal logarithmic increase in the *pp* cross sections of the form $(\ln s)^2$. We note here that a $\ln^2$ relation implies a *twin* prime–index quotient according to the Brun theorem. Viggo Brun showed that the sum of reciprocals of the twin primes was convergent [14]. The *Brun*'*s* argument can be used to show that the number of twin primes less than N does not exceed $\frac{CN}{(\ln N)^2}$ and paved the way for the Hardy–Littlewood relation [15].

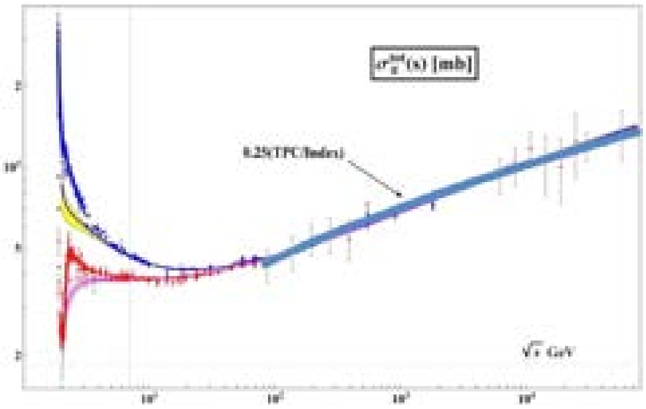

Figure 4. The visible energy distribution for proton and antiprotons. The total cross sections, i.e. the ratio of TPC to index have been multiplied by 0.25 for normalizing to the experimental data.

$$\pi_2(x) \sim 2C_2 \frac{x}{(\ln x)^2}. \tag{10}$$

In equation (10) π_2 is the index of a twin prime pair and x is a pair of twin primes. Hence, we can write the $(\ln x)^2$ as the ratio of a twin prime to its index. We generated the first 6.0×10^8 twin primes by using their *companion* which we refer to as the Twin Prime Companion or TPC [16, 17]. A TPC is the composite sandwiched between a pair of twin primes. As shown in figure 4, the ratio of the TPC to its index is the pp cross section in mb and the index is the energy in GeV. Note, the ratio of TPC to the index has been multiplied by 0.25 to normalize it to the experimental data.

6. Summary

In summary, the total neutrino and antineutrno cross sections follow very closely the increase in the magnitude of primes as a function of their positional indices. Neutrino and antineutrino nucleon total CC cross sections are estimated by the magnitude of the prime numbers over six (6) decades of neutrino energy where experimental data are available. In this proposed method, the cross sections are obtained from a two-column look-up table where the first column is the index of the prime and it represents the neutrino energy in MeV and the second column is the prime number multiplied by 0.70. This represents the neutrino-nucleon cross section in *ab*. For antineutrinos, the second column is multiplied by 0.26 to reproduce the cross sections in *ab*. At very high energies, the prime–index method tends to indicate increases in neutrino-nucleon cross sections consistent with observed IceCube data. Note, the prime–index method also provides a quick and reliable rate estimate for various neutrino experiments spanning 6 decades of neutrino energies. With the available published astrophysical muon neutrinos the prime–index method predicts 11 ± 3.3 events versus the observed 9 ± 3 events. Note, the SM calculations predict 1 ± 1 event only. The method was also applied to the IceCube cascade data which represents a mixture of CC and NC. A similar method, based on *twin primes* and their indices, has been employed for high energy pp cross sections which explains the $(\ln s)^2$ parametrization first proposed by *Heisenberg*.

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